Teori Himpunan

Bagian III

Teori Himpunan

- Himpunan: Kumpulan objek (konkrit atau abstrak) yang mempunyai syarat tertentu dan jelas, bisanya dinyatakan dengan huruf besar.
- $a \in A$ "a anggota dari *A*"
- $a \notin A$ "a bukan anggota dari *A*"
- $A = \{a_1, a_2, \ldots, a_n\}$ "*A* memuat…"

Cara menyatakan himpunan

a. Mendaftar
b. Menyatakan sifat-sifat yang dipenuhi oleh anggota.
c. Notasi pembentuk himpunan

Notasi Pembentuk Himpunan

Format: "sedemikian hingga"

{[struktur keanggotaan] | [syarat perlu untuk menjadi anggota]}

Contoh:

$Q = \{m/n : m,n \in Z, n \neq 0\}$

- Q adalah himpunan bilangan rasional
- Elemen-elemennya berstruktur m/n; harus memenuhi sifat setelah tanda ":" untuk menjadi anggota.

$\{x \in R \mid x^2 = 1\} = \{-1,1\}$

Contoh Himpunan:

N – himpunan bil. Cacah = {0,1,2,3,4, …}

P atau Z+ - himp. Bil. Bulat positif = {1,2,3,4, …}

Z – himpunan bil. bulat

R – himpunan bil. real

ϕ or {} – himpunan kosong

U – himpunan semesta, himp. yang memuat semua element yang dibicarakan.

Contoh Himpunan

- $A = \emptyset$ "empty set/null set"
- $A = \{z\}$ Note: $z \in A$, but $z \neq \{z\}$
- $A = \{\{b, c\}, \{c, x, d\}\}$
- $A = \{\{x, y\}\}$
  Note: $\{x, y\} \in A$, but $\{x, y\} \neq \{\{x, y\}\}$
- $A = \{x \mid P(x)\}$
  "set of all x such that P(x)"
- $A = \{x \mid x \in \mathbf{N} \wedge x > 7\} = \{8, 9, 10, \ldots\}$
  "set builder notation"

Relasi Antar Himpunan

1. Himpunan yang Sama
2. Himpunan Bagian
3. Himpunan yang berpotongan
4. Himpunan Saling Lepas
5. Himpunan yang Ekuivalen

Himpunan yang Sama
(Set Equality)

Himp. A and B dikatakan sama jika keduanya memuat anggota-anggota yang tepat sama.

$A = B \Leftrightarrow \{ x \mid x \in A \leftrightarrow x \in B\}$ atau $A = B \Leftrightarrow A \subset B \wedge B \subset A$

Contoh:

- A = {9, 2, 7, -3}, B = {7, 9, -3, 2} : $A = B$
- A = {dog, cat, horse},
  B = {cat, horse, squirrel, dog} : $A \neq B$
- A = {dog, cat, horse},
  B = {cat, horse, dog, dog} : $A = B$

Himpunan Bagian

$A \subseteq B$ "A adalah himpunan bagian dari B"

$A \subseteq B$ jika setiap anggota A juga merupakan anggota B.

$$A \subseteq B \Leftrightarrow \forall x \, (x \in A \rightarrow x \in B)$$

Contoh:

$A = \{3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? benar

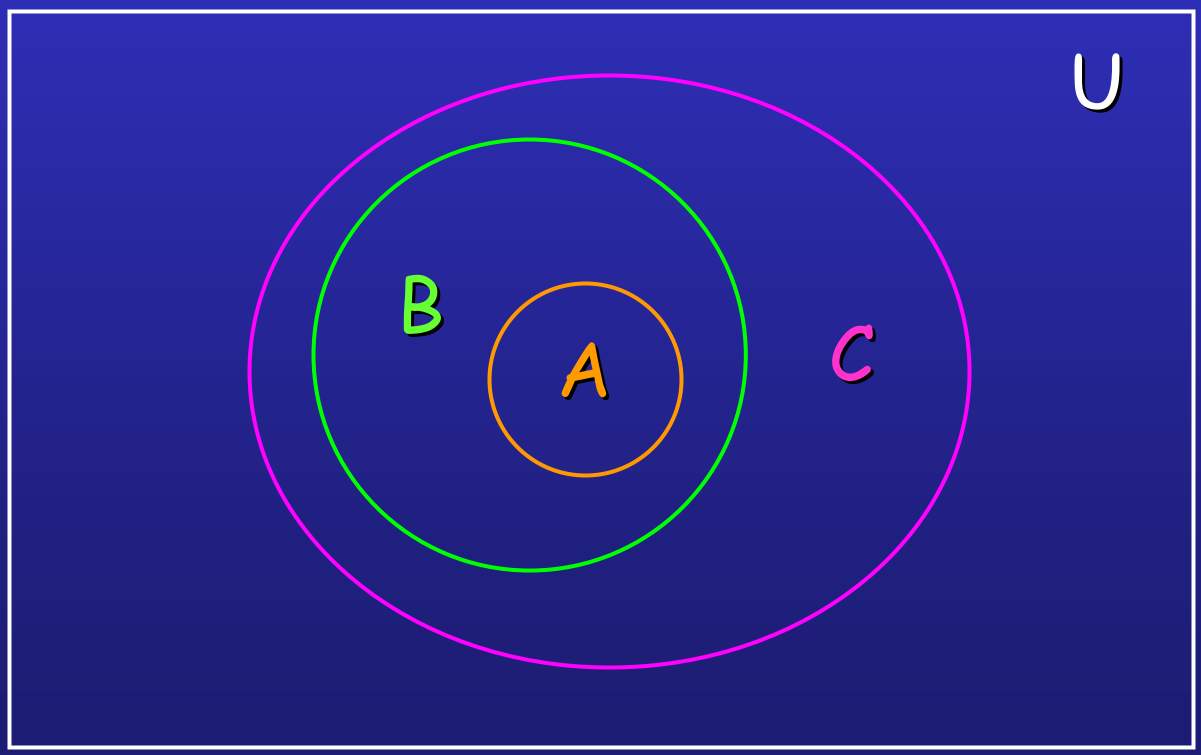
$A = \{3, 3, 3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? benar

$A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $A \subseteq B$? Salah

Himpunan Bagian

Sifat:

- $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$
- $(A \subseteq B) \wedge (B \subseteq C) \Rightarrow A \subseteq C$ (Lihat Venn Diagram)



Himpunan Bagian

Useful rules:

- $\emptyset \subseteq A$ for any set A
- $A \subseteq A$ for any set A

Proper subsets (Himpunan Bagian Sejati):

$A \subset B$ "A is a proper subset of B"

$A \subset B \Leftrightarrow \forall x\,(x \in A \rightarrow x \in B) \wedge \exists x\,(x \in B \wedge x \notin A)$

or

$A \subset B \Leftrightarrow \forall x\,(x \in A \rightarrow x \in B) \wedge \neg\forall x\,(x \in B \rightarrow x \in A)$

Dua himpunan A dan B dikatakan berpotongan, ditulis A)(B, jika ada anggota A yang menjadi anggota B.

$A)(B \Leftrightarrow \exists x\ (x \in A \wedge x \in B)$

Himpunan A dan B dikatakan saling lepas (A//B), jika $A \neq \emptyset$, $B \neq \emptyset$, $\forall x\ (x \notin A \vee x \notin B)$

Himpunan A dan B yang Ekuivalen, A~B, jika setiap anggota A dapat dipasangkan (dikorespondensikan) satu-satu dengan anggota B

Buat Contoh Masing-masing!!!

Latihan

1. Buktikan jika $M \subset \emptyset$, maka $M = \emptyset$.
2. $A = \{1,2,3,4\}$; B = himpunan bilangan ganjil. Buktikan $A \not\subset B$.
3. Buktikan $A \subset B, B \subset C \rightarrow A \subset C$.
4. Buktikan $K \subset L, L \subset M, M \subset K \rightarrow K = M$.

Interval Notation - Special notation for subset of R

$[a,b] = \{x \in R \mid a \leq x \leq b\}$

$(a,b) = \{x \in R \mid a < x < b\}$

$[a,b) = \{x \in R \mid a \leq x < b\}$

$(a,b] = \{x \in R \mid a < x \leq b\}$

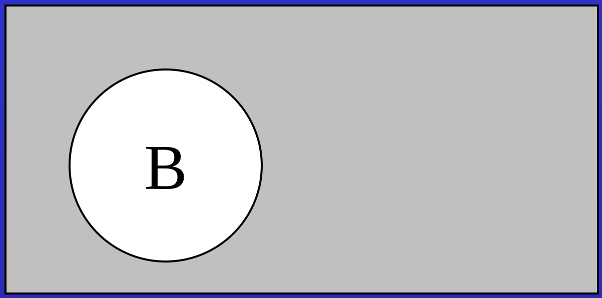
How many elements in [0,1]?

In (0,1)?

In {0,1}

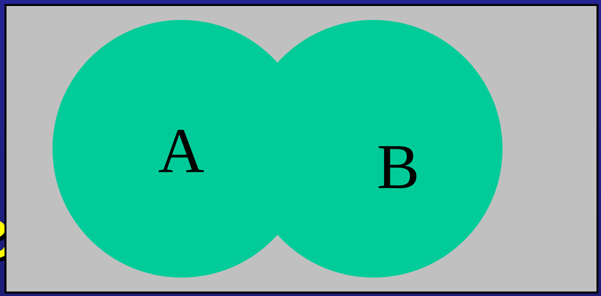
Operasi Himpunan

$\overline{B}$ (B complement)



- $\{x \mid x \in U \wedge x \notin B\}$
- Everything in the Universal set that is not in B

$A \cup B$ (A union B)



- $\{x \mid x \in A \vee x \in B\}$
- Like inclusive or, can be in A or B or both

$A \cap B$ (A intersect B)

- $\{x \mid x \in A \wedge x \in B\}$
- A and B are disjoint if $A \cap B = \Phi$

A - B (A minus B or difference)

- $\{x \mid x \in A \wedge x \notin B\}$
- $A - B = A \cap \overline{B}$

$A \oplus B$ (symmetric difference)

- $\{x \mid x \in A \oplus x \in B\} = (A \cup B) - (A \cap B)$
- We have overloaded the symbol $\oplus$. Used in logic to mean exclusive or and in sets to mean symmetric difference

Contoh

Let $A = \{n^2 \mid n \in P \wedge n \leq 4\} = \{1,4,9,16\}$

Let $B = \{n^4 \mid n \in P \wedge n \leq 4\} = \{1,16,81,256\}$

$A \cup B = \{1,4,9,16,81,256\}$

$A \cap B = \{1,16\}$

$A - B = \{4,9\}$

$B - A = \{81, 256\}$

$A \oplus B = \{4,9,81,256\}$

Cardinality of Sets

If a set S contains n distinct elements, $n \in \mathbf{N}$, we call S a finite set with cardinality n.

Examples:

$A = \{\text{Mercedes, BMW, Porsche}\}$, $|A| = 3$

$B = \{1, \{2, 3\}, \{4, 5\}, 6\}$ $|B| = 4$

$C = \emptyset$ $|C| = 0$

$D = \{ x \in \mathbf{N} \mid x \leq 7000 \}$ $|D| = 7001$

$E = \{ x \in \mathbf{N} \mid x \geq 7000 \}$ E is infinite!

The Power Set

P(A) "power set of A"

$P(A) = \{B \mid B \subseteq A\}$ (contains all subsets of A)

Examples:

$A = \{x, y, z\}$

$P(A) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$

$A = \emptyset$

$P(A) = \{\emptyset\}$

Note: $|A| = 0$, $|P(A)| = 1$

The Power Set

Cardinality of power sets:

$| P(A) | = 2^{|A|}$

- Imagine each element in A has an "on/off" switch
- Each possible switch configuration in A corresponds to one element in 2^A

A	1	2	3	4	5	6	7	8
x	x	x	x	x	x	x	x	x
y	y	y	y	y	y	y	y	y
z	z	z	z	z	z	z	z	z

- For 3 elements in A, there are $2 \times 2 \times 2 = 8$ elements in P(A)

Cartesian Product

The ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ is an ordered collection of objects.

Two ordered n-tuples $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ are equal if and only if they contain exactly the same elements in the same order, i.e. $a_i = b_i$ for $1 \leq i \leq n$.

The Cartesian product of two sets is defined as:

$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Example: $A = \{x, y\}$, $B = \{a, b, c\}$

$A \times B = \{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)\}$

Cartesian Product

The Cartesian product of two sets is defined as:

$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Example:

$A = \{good, bad\}$, $B = \{student, prof\}$

$A \times B = \{(good, student), (good, prof), (bad, student), (bad, prof)\}$

$B \times A = \{(student, good), (prof, good), (student, bad), (prof, bad)\}$

Cartesian Product

Note that:

- $A \times \emptyset = \emptyset$
- $\emptyset \times A = \emptyset$
- For non-empty sets A and B: $A \neq B \Leftrightarrow A \times B \neq B \times A$
- $|A \times B| = |A| \cdot |B|$

The Cartesian product of two or more sets is defined as:

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } 1 \leq i \leq n\}$

Set Operations

Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$

$A \cup B = \{a, b, c, d\}$

Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$

$A \cap B = \{b\}$

Set Operations

Two sets are called disjoint if their intersection is empty, that is, they share no elements:

$A \cap B = \emptyset$

The difference between two sets A and B contains exactly those elements of A that are not in B:

$A-B = \{x \mid x \in A \land x \notin B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$, $A-B = \{a\}$

Set Operations

The complement of a set A contains exactly those elements under consideration that are not in A:

$A^c = U - A$

Example: $U = \mathbf{N}$, $B = \{250, 251, 252, \ldots\}$

$B^c = \{0, 1, 2, \ldots, 248, 249\}$

Set Operations

Table 1 in Section 1.5 shows many useful equations.

How can we prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$?

Method I:

$x \in A \cup (B \cap C)$

$\Leftrightarrow x \in A \lor x \in (B \cap C)$

$\Leftrightarrow x \in A \lor (x \in B \land x \in C)$

$\Leftrightarrow (x \in A \lor x \in B) \land (x \in A \lor x \in C)$

(distributive law for logical expressions)

$\Leftrightarrow x \in (A \cup B) \land x \in (A \cup C)$

$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$

Set Operations

Method II: Membership table

1 means "x is an element of this set"

0 means "x is not an element of this set"

A B C	B∩C	A∪(B∩C)	A∪B	A∪C	(A∪B) ∩(A∪C)
0 0 0	0	0	0	0	0
0 0 1	0	0	0	1	0
0 1 0	0	0	1	0	0
0 1 1	1	1	1	1	1
1 0 0	0	1	1	1	1
1 0 1	0	1	1	1	1
1 1 0	0	1	1	1	1
1 1 1	1	1	1	1	1

Sifat Operasi Himpunan

1. Asosiatif: $(A \cup B) \cup C = A \cup (B \cup C)$
   $(A \cap B) \cap C = A \cap (B \cap C)$
2. Idempoten: $A \cup A = A$; $A \cap A = A$
3. Identitas: $A \cup S = S$; $A \cap S = A$
   $A \cup \varnothing = A$; $A \cap \varnothing = \varnothing$
4. Distributif: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. Komplementer: $A \cup A' = S$; $A \cap A' = \varnothing$
6. De Morgan: $(A \cup B)' = A' \cap B'$
   $(A \cap B)' = A' \cup B'$
7. Penyerapan: $A \cup (A \cap B) = A$
   $A \cap (A \cup B) = A$

Latihan

1. Buktikan $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. Buktikan $A - (B \cup C) = (A - B) \cap (A - C)$
3. Bila $A \subset B$, buktikan $A \cap B = A$ dan $A \cup B = B$
4. Buktikan $(A \cup B) \times C = (A \times C) \cup (B \times C)$